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Abstract
The symplectic geometry of the phase space associated with a charged particle is determined by the addition of the Faraday 2-form to the standard $dp \wedge dq$ structure on $\mathbb{R}^{2n}$. In this paper, we describe the corresponding algebra of Weyl-symmetrized functions in operators $\hat{q}$, $\hat{p}$ satisfying nonlinear commutation relations. The multiplication in this algebra generates an associative $*$ product of functions on the phase space. This $*$ product is given by an integral kernel whose phase is the symplectic area of a groupoid-consistent membrane. A symplectic phase space connection with non-trivial curvature is extracted from the magnetic reflections associated with the Stratonovich quantizer. Zero and constant curvature cases are considered as examples. The quantization with both static and time-dependent electromagnetic fields is obtained. The expansion of the $*$ product by the deformation parameter $\hbar$, written in the covariant form, is compared with the known deformation quantization formulae.

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1. Introduction

An associative non-commutative multiplication of functions on phase space, corresponding to the Poisson structure, is called a *quantization*. On general phase spaces, i.e. on symplectic manifolds having a symplectic connection, there is now a well-developed scheme of deformation quantization [1–6]. This fundamental theory is formulated in symplectic terms, but is based on formal asymptotic expansions and generally does not have an operator representation in Hilbert space.

Only a few examples are known where the quantization is *perfect*, that is:

- it is exact, rather than expressed via asymptotic series;
- it has an operator representation in a Hilbert space (corresponding to Schrödinger quantum mechanics); and
- it is given explicitly and purely in symplectic terms.

The first two conditions from this list are realized in certain examples of the strict deformation quantization [7, 8] and in the tangential groupoid quantization [9, 10], but the last (geometrical) criterion is not fulfilled.

The only known perfect examples are related to the phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ with canonical symplectic structure and the trivial connection, or to cylinder-type spaces, where the coordinates are subject to periodicity conditions [11], or to a generic symplectic form but with a flat (zero curvature) symplectic connection [12, 13]. We do not refer here to the homogeneous Kählerian spaces where the quantization by coherent states could be considered as perfect, but involves symplectic areas in a complexified phase space.

The example of perfect quantization which we construct in the present paper is concerned with non-zero and non-constant phase space curvature. The formalism allows one to represent, in a manifestly gauge invariant and covariant manner, the dynamics of a charged particle in an electromagnetic field realized in terms of a quantum phase space.

There are two ways to introduce the magnetic coupling into quantum mechanics. The first is based on the interpretation of the magnetic potential as a connection form in the $U(1)$-bundle over the configuration space and subsequently considers the corresponding modification of the dynamical (Schrödinger, Klein–Gordon, etc) equations. The second approach incorporates the idea of modifying the usual symplectic form $\mathrm{d}p \wedge \mathrm{d}q$ on phase space by adding the Faraday 2-form and then quantizing this new symplectic space, and in particular, representing functions on this space by operators. The present paper employs this second method.

We consider the phase space $\mathbb{R}^{2n} = \mathbb{R}_q^n \oplus \mathbb{R}_p^n$ with the following 'magnetic' symplectic form

$$\omega = \mathrm{d}p \wedge \mathrm{d}q + \tfrac{1}{2}F(q)\,\mathrm{d}q \wedge \mathrm{d}q. \tag{1.1}$$

The coordinates p have the physical interpretation of the gauge invariant (kinetic) momenta and F is a skew tensor on the configuration space $\mathbb{R}_q^n$ representing the magnetic portion of the electromagnetic field, cf [14]. The closedness of form (1.1) is equivalent to the homogeneous Maxwell equation for the Faraday tensor F. The charge coupling constant and the speed of light are all set equal to 1.

The non-degenerate symplectic form (1.1) is a simple modification of the canonical form $\mathrm{d}p \wedge \mathrm{d}q$, but nevertheless the appearance of a generic tensor F makes the structure of the quantum phase space rather non-trivial. In general, we assume that the components of the tensor $F(q)$ are nonlinear functions of q, but note that the linear and constant cases still provide interesting physical examples.

Our procedure for constructing the quantum phase space is the following. First, we quantize the Poisson brackets related to the form ω and immediately obtain the commutation relations between the quantum coordinates

$$[\hat{q}^j, \hat{q}^s] = 0 \qquad [\hat{q}^j, \hat{p}_s] = \mathrm{i}\hbar\delta_s^j \qquad [\hat{p}_j, \hat{p}_s] = \mathrm{i}\hbar F_{sj}(\hat{q}). \tag{1.2}$$

The nonlinearity of F in the momentum commutation relations means that Lie algebra techniques are not applicable here. Furthermore, as we shall see, it is precisely this nonlinearity that is responsible for the appearance of quantum phase space curvature.

It is easy to represent relations (1.2) in terms of self-adjoint operators on $L^2(\mathbb{R}^n)$ and then to construct the Weyl-symmetrized functions of those operators, specifically

$$\hat{f} = f(\hat{q}, \hat{p}) = f\left(\frac{\overset{1}{\hat{q}} + \overset{3}{\hat{q}}}{2}, \overset{2}{\hat{p}}\right). \tag{1.3}$$

The over numbering of the operators indicates the order in which they act on a target wavefunction as in [15]. Thus, we have a linear mapping $f \to \hat{f}$. Considering this mapping as an operator-valued linear functional, we can represent it in the integral form

$$\hat{f} = \int f(x) \Delta(x) \, dx. \tag{1.4}$$

Here $x = (q, p)$ denote phase space points. In this way, we shall obtain a family of operators (quantizers) $\Delta(x)$ acting in the same Hilbert space where the representation of algebra (1.2) is given. De-quantization, the inverse map to (1.4), is also constructed by the quantizer. We shall see that for a suitable class of operators,

$$f(x) = (2\pi\hbar)^n \operatorname{tr}(\hat{f} \Delta(x)). \tag{1.5}$$

We say that f is the (magnetic) *symbol* image of the operator $\hat{f}$.

The family Δ in the case of zero magnetic tensor was first introduced in [16] and has since been intensively studied for algebras with linear commutation relations in [10, 13, 17].

For general nonlinear tensors F in (1.2) the quantizer Δ is still well defined and possesses the following basic properties:

 (i) the elements of Δ are linearly independent, invertible and resolve the identity;
(ii) the linear envelope of elements of Δ forms an algebra.

The 'structure constants' of this algebra generate a non-commutative $*$ product of functions on phase space,

$$\widehat{f * g} = \hat{f} \hat{g}. \tag{1.6}$$

We call $*$ a *magnetic product*. This product satisfies the correspondence principle:

$$f * g = fg - \frac{i\hbar}{2}\{f, g\} + O(\hbar^2) \qquad \text{as} \quad \hbar \to 0$$

(as is usual in the deformation quantization scheme), where $\{\cdot, \cdot\}$ denotes the Poisson bracket related to the symplectic form (1.1) and fg is the commutative product of functions.

Our main goal is to interpret this magnetic product geometrically, and to demonstrate how the quantizer generates a phase space connection.

First, we shall see that the quantizer $\Delta(z)$ generates a symplectic transformation $\sigma_z : x \to x'$ in $\mathbb{R}^{2n}$. This transformation is given by the Fock-type formula [18]:

$$\Delta(z)^{-1} \hat{x} \Delta(z) = \hat{x}'. \tag{1.7}$$

Here $\hat{x} = (\hat{q}, \hat{p})$ is the set of generators appearing in (1.2), and $\hat{x}' = (\hat{q}', \hat{p}')$ is a new set (with the same commutation relations). The symbol image of (1.7) defines σ_z.

For each $z \in \mathbb{R}^{2n}$ the mapping $\sigma_z : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserves the symplectic form ω, has the fixed point $z = \sigma_z(z)$, and is an involution: $\sigma_z^2 = \mathrm{id}$. We call $\sigma = \{\sigma_z\}$ a family of *magnetic reflections*. Using these reflections one can realize the symplectic groupoid multiplication rule corresponding to relations (1.2) [19, 20]. Then for each triplet of points z, y, x we can construct a membrane $\Sigma(z, y, x)$ in $\mathbb{R}^{2n}$ whose boundary is consistent with the groupoid structure; namely a boundary consisting of three linked σ-reflective curves with mid-points z, y, x.

Based on our previous results [19, 21], we shall obtain the following formula for the magnetic non-commutative product:

$$(f * g)(z) = \frac{1}{(\pi\hbar)^{2n}} \iint \exp\left\{\frac{i}{\hbar} \int_{\Sigma(z,y,x)} \omega\right\} f(y) g(x) \, dy \, dx. \tag{1.8}$$

So, we see that the membrane WKB phase of the $*$ product integral kernel, conjectured in [22] for symmetric symplectic manifolds, is realized in the magnetic phase space exactly, without the need for a WKB expansion.

The product (1.8) is strict (not formal). Its asymptotic expansion as $\hbar \to 0$ can be written in the bi-differential covariant form

$$f * g = fg - \frac{i\hbar}{2} f \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle g - \frac{\hbar^2}{8} f \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^2 g + O(\hbar^3). \tag{1.9}$$

Here $\Psi = \begin{bmatrix} 0 & -I \\ I & F \end{bmatrix}$ is the Poisson tensor corresponding to the symplectic structure ω. The covariant derivative ∇ acts either on the left multiplier or the right multiplier as indicated by the arrows but does not act on the argument of Ψ. The derivative above corresponds to a connection on the phase space $\mathbb{R}^{2n}$ defined by the following Christoffel symbols:

$$\Gamma_{sl}^j(x) = -\frac{1}{2} \frac{\partial^2 \sigma_z(x)^j}{\partial x^s \partial x^l} \Big|_{z=x}. \tag{1.10}$$

We call this a *magnetic connection*. It is symplectic: $\nabla \omega = 0$. We emphasize that this phase space connection is generated by the tensor F given on configuration space $\mathbb{R}_q^n$, but Γ is not a Riemannian-type connection.

All the higher terms $O(\hbar^k)$ in (1.9) contain the Poisson bracket contribution $f \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^k g$ plus additional terms generated by the curvature of Γ, so that

$$f * g = f \exp \left\{ -\frac{i\hbar}{2} \langle \overleftarrow{\nabla} \Psi(x) \overrightarrow{\nabla} \rangle + \frac{i\hbar^3}{48} R^{ijkl} (\overrightarrow{\nabla}_i \overrightarrow{\nabla}_j \overrightarrow{\nabla}_k \overleftarrow{\nabla}_l - \overleftarrow{\nabla}_i \overleftarrow{\nabla}_j \overleftarrow{\nabla}_k \overrightarrow{\nabla}_l) + O(\hbar^4) \right\} g \tag{1.11}$$

where R is the curvature tensor with all indices raised by the Poisson structure.

If the tensor F is quadratic in Euclidean coordinates on $\mathbb{R}_q^n$, then the curvature tensor R on the phase space $\mathbb{R}^{2n} = T^* \mathbb{R}^n$ is constant but non-zero. This case provides a rather interesting example of a symmetric symplectic space (in the terminology of [23]). Here the mappings σ possess an additional property

$$\sigma_y \sigma_z \sigma_y(x) = \sigma_{\sigma_y(z)}(x) \qquad \forall\, x, y, z \tag{1.12}$$

and the connection (1.10) is recognized as the Cartan–Loos canonical connection [24] corresponding to the family of σ-symmetries.

In the Riemannian setting, Cartan [25] called this class of spaces 'remarkable'. In our magnetic framework this symmetric structure does not belong to Riemannian geometry but still is remarkable. The commutation relations (1.2) in this case look quadratic

$$[\hat{q}^j, \hat{q}^s] = 0 \qquad [\hat{q}^j, \hat{p}_s] = i\hbar \delta_s^j \qquad [\hat{p}_j, \hat{p}_s] = i\hbar F_{sj,kl} \hat{q}^k \hat{q}^l. \tag{1.13}$$

The integral formula (1.8) for the associative product corresponding to this quadratic relations employs membranes $\Sigma(z, y, x)$ bounded by just geodesics of the Cartan–Loos connection. In the bi-differential formula (1.11) the $O(\hbar^4)$ remainder vanishes in this case.

So, in summary, the new features developed in the paper are:

- construction of a phase space connection and curvature generated by a generic (electro) magnetic tensor;
- geometric groupoid interpretation of membrane areas in the integral formula for the associative product corresponding to the nonlinear commutation relations (1.2);

- realization of the symmetric symplectic structure related to quadratic brackets (1.13) and its explicit quantization by means of geodesically bounded membranes or by the curvature generated bi-differential exponent.

This paper is organized as follows. Section 2 describes representations of the quantizer and the construction of the reflection map σ_x. The magnetic $*$ product and its groupoid aspects are discussed in section 3. The magnetic connection is found in section 4 and the curvature features of the $\hbar$ deformation expansion the $*$ product are presented there. The role of the electric field is clarified in section 5. The next two sections treat the zero-curvature and constant-curvature cases.

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2. Quantizer and magnetic reflections

First we make a general remark about the operator calculations made below. All of them are simple direct constructions and all the formulae are obtained explicitly, although from the view point of functional analysis, the presentation of results often looks formal. But actually the suppressed functional analysis details are standard (about this, see the remarks at the end of section 3) and of limited usefulness in clarifying the new objects and results coming out of the calculus. Of course, it is known that the use of formal methods in non-commutative analysis (even for algebras with the simplest Heisenberg commutation relations) can lead to errors. A list of problems demonstrating the 'dangerous' areas where the formal analysis gives incorrect results is found in the book [20], appendix 1. However, the derivations below are far from these sensitive analytical areas.

In this section, we review the definition of the magnetic quantizer and introduce the associated reflective structure.

The irreducible representation of commutation relations (1.2) in the Hilbert space $L^2(\mathbb{R}^n)$ is given by the operators

$$\hat{q}^j : \psi(q') \mapsto q'^j \psi(q') \qquad \hat{p}_j : \psi(q') \mapsto -i\hbar \frac{\partial \psi(q')}{\partial q'^j} - A_j(q')\psi(q'). \tag{2.1}$$

Here q' is running over $\mathbb{R}^n$, and A_j are components of the 1-form $\mathcal{A} = A_j(q')\,dq'^j$ which is a primitive of the Faraday 2-form $\frac{1}{2}F_{jk}(q')\,dq'^k \wedge dq'^j$, namely

$$\frac{\partial A_j(q')}{\partial q'^k} - \frac{\partial A_k(q')}{\partial q'^j} = F_{jk}(q'). \tag{2.2}$$

The operators (2.1) are well defined on a dense domain in $L^2(\mathbb{R}^n)$ and essentially self-adjoint. Thus one can consider Weyl-symmetrized functions of these operators following the general definitions in [26, 27, 20]. In detail, we take smooth and rapidly decaying functions $f = f(x)$, introduce their Fourier transform $\tilde{f}$ and obtain operators in $L^2(\mathbb{R}^n)$ via

$$\hat{f} = \int \tilde{f}(\eta) \exp\left\{ \frac{i}{\hbar} \eta \hat{x} \right\} d\eta \qquad \hat{x} = (\hat{q}, \hat{p}). \tag{2.3}$$

It is easy to see that

$$\exp\left\{ \frac{i}{\hbar} \eta \hat{x} \right\} = \exp\left\{ \frac{i}{2\hbar} \eta_q \hat{q} \right\} \exp\left\{ \frac{i}{\hbar} \eta_p \hat{p} \right\} \exp\left\{ \frac{i}{2\hbar} \eta_q \hat{q} \right\} \tag{2.4}$$

where η_q and η_p are just the components of the vector $\eta \in \mathbb{R}^n_q \oplus \mathbb{R}^n_p$.

From factorization (2.4) we observe that formula (1.3) follows. Also, from the definition of momentum operators (2.1) we have

$$\exp\left\{\frac{i}{\hbar}\eta_p\hat{p}\right\} = \exp\left\{\eta_p\frac{\partial}{\partial q'} - \frac{i}{\hbar}\eta_p A(q')\right\}$$

$$= \exp\left\{\eta_p\frac{\overset{2}{\partial}}{\partial q'} - \frac{i}{\hbar}\int_0^1 \eta_p A((1-\tau)\overset{3}{q'} + \tau\overset{1}{q'})\mathrm{d}\tau\right\}$$

$$= \exp\left\{-\frac{i}{\hbar}\int_0^1 \eta_p A((1-\tau)q' + \tau(q'+\eta_p))\,\mathrm{d}\tau\right\}\exp\left\{\eta_p\frac{\partial}{\partial q'}\right\}.$$

So, on any function $\psi \in L^2(\mathbb{R}^n)$ this operator exponential acts as follows:

$$\exp\left\{\frac{i}{\hbar}\eta_p\hat{p}\right\}\psi(q') = \exp\left\{-\frac{i}{\hbar}\int_0^1 \eta_p A(q'+\tau\eta_p)\,\mathrm{d}\tau\right\}\psi(q'+\eta_p)$$

$$= \exp\left\{-\frac{i}{\hbar}\int_{q'}^{q'+\eta_p} \mathcal{A}\right\}\psi(q'+\eta_p)$$

where the integral of the 1-form $\mathcal{A}$ is taken along the straight-line segment in $\mathbb{R}^n$ connecting q' to $q'+\eta_p$.

Thus formula (2.4) implies

$$\exp\left\{\frac{i}{\hbar}\eta\hat{x}\right\}\psi(q') = \exp\left\{\frac{i}{2\hbar}\eta_q\eta_p - \frac{i}{\hbar}\int_{q'}^{q'+\eta_p} \mathcal{A} + \frac{i}{\hbar}\eta_q q'\right\}\psi(q'+\eta_p). \tag{2.5}$$

Taking the inverse Fourier transform of (2.3) and representing the operators $\hat{f}$ in the form (1.4) we obtain the quantizer acting on the function ψ,

$$\Delta(q,p)\psi(q') = \frac{1}{(\pi\hbar)^n}\exp\left\{\frac{2i}{\hbar}p(q'-q) + \frac{i}{\hbar}\int_{2q-q'}^{q'} \mathcal{A}\right\}\psi(2q-q'). \tag{2.6}$$

Observe that the point $2q - q'$ in the rightmost ψ is just the original q' reflected through the point q with respect to the Euclidean structure on $\mathbb{R}^n$. The integral kernel statement equivalent to (2.6) is

$$\langle q'|\Delta(q,p)|q''\rangle = (2\pi\hbar)^{-n}\delta\left(\frac{q'+q''}{2} - q\right)\exp\left\{\frac{i}{\hbar}p(q'-q'') + \frac{i}{\hbar}\int_{q''}^{q'} \mathcal{A}\right\}. \tag{2.7}$$

The delta function forces the value of the midpoint $(q'+q'')/2$ to be q. The representations (2.6) and (2.7) are similar to those used by Stratonovich [28] in defining the gauge invariant Wigner transform.

Using the above representations it is easy to verify the following properties of the family of operators $\Delta(x)$ $(x = (q,p) \in \mathbb{R}^{2n})$ acting in the Hilbert space $L^2(\mathbb{R}^n)$.

Lemma 1.

(i) $\int \Delta(x)\,\mathrm{d}x = I$;

(ii) $\Delta(x)^\dagger = \Delta(x)$, $\Delta(x)^2 = \frac{1}{(\pi\hbar)^{2n}}I$;

(iii) $\operatorname{tr}\Delta(x_1)\Delta(x_2) = (2\pi\hbar)^{-n}\delta(x_1 - x_2)$, $\operatorname{tr}\Delta(x) = (2\pi\hbar)^{-n}$.

The trace used above is understood in the distributional sense, so that the generalized functions $\operatorname{tr}\Delta(x)$ and $\operatorname{tr}(\Delta(y)\Delta(x))$ are naturally defined as distributions on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ respectively, that is the operator-valued functions Δ are first integrated in the Bochner sense

with test functions and after that the trace operation is applied. The operator identities and equations above and throughout the text are considered in the sense of the strong topology on a dense domain and then extended (where possible) to the whole Hilbert space $L^2(\mathbb{R}^n)$.

Equation (i) in lemma 1 says that $\{\Delta(x)\}$ is a resolution of the identity; as an example of (1.4) it states that the unit symbol $f(x) = 1$ is quantized to the identity operator, $\hat{f} = I$. Property (ii) shows that $\Delta(x)$ are bounded self-adjoint operators in $L^2(\mathbb{R}^n)$ with norm $(\pi\hbar)^{-n}$; modulo a rescaling, $\Delta(x)$ is the unitary operator. The first part of (iii) in combination with (1.4) establishes that the de-quantization map $\hat{f} \to f$ is given by the trace identity (1.5) and, as a consequence, that the symbol of $\Delta(x)$ is the delta function δ_x.

We now consider the construction of the reflection map induced by $\Delta(x)$. Given (2.6) the intertwining identities readily follow:

$$\begin{aligned} \hat{q}^j \Delta(q, p) &= \Delta(q, p)(2q - \hat{q})^j \\ \hat{p}_j \Delta(q, p) &= \Delta(q, p)(2p - \hat{p} - \hat{\alpha}_q)_j \end{aligned} \qquad j = 1, \ldots, n. \qquad (2.8)$$

Here α_q is the following vector function:

$$\alpha_q(q') = A(q') + A(2q - q') - \frac{\partial}{\partial q'}\left(\int_{2q-q'}^{q'} \mathcal{A}\right). \qquad (2.9)$$

A simple calculation shows one can restate this in a gauge invariant fashion as the average of the magnetic tensor F,

$$\alpha_q(q') = \int_{-1}^{1} F(q + \mu(q' - q))(q' - q)\mu \, d\mu. \qquad (2.10)$$

Left multiply (2.8) by $\Delta(x)^{-1}$ and thereby obtain $x' \to \sigma_x(x')$ as

$$\sigma_x(x') = 2x - x' - \begin{pmatrix} 0 \\ \alpha_{x_q}(x'_q) \end{pmatrix} \qquad (2.11)$$

where x_q and x'_q denote the q-components of the phase space points x and x'.

Lemma 2. *The family of mappings $\{\sigma_x\}$ in the magnetic phase space $\prec \mathbb{R}^{2n}, \omega \succ$ possesses the following properties:*

(i) $\sigma_x(x) = x, \forall x$;
(ii) $\sigma_x^2 = \mathrm{id}$;
(iii) σ_x is symplectic, i.e. preserves the magnetic 2-form ω.

Proof. Part (i) follows from $\alpha_q(q) = 0$, and (ii) from $\alpha_q(2q - q') = \alpha_q(q')$. Property (iii) is a result of the fact that the operators $\hat{x}' = \hat{\sigma}_z$ and $\hat{x}$ satisfy the same commutation relations (1.2). So the magnetic 2-form ω must be invariant under the change of variables $x' \to \sigma_x(x')$ $\qquad\qquad\square$

Property (ii) is the symbol equivalent of the involution identity, $\Delta(x)^2 = (\pi\hbar)^{-2n} I$. Also note that the pullback invariance, $\sigma_x^* \omega = \omega$, means that the vector function α_q satisfies the tensor identity,

$$\frac{\partial \alpha_q(q')_j}{\partial q'^k} - \frac{\partial \alpha_q(q')_k}{\partial q'^j} = F_{jk}(q') - F_{jk}(2q - q') \qquad (2.12)$$

which easily follows from (2.9).

The presence of reflective transformations on phase space allows one to identify a useful family of curves. A continuous, piecewise differentiable function $x : [-t, t] \mapsto \mathbb{R}^{2n}$ is called a *magnetic reflective curve* with midpoint $x(0)$ if

$$\sigma_{x(0)}(x(\tau)) = x(-\tau) \qquad \tau \in [-t, t].$$

Clearly a reflective curve has a central symmetry about its midpoint. These curves will be used to define the boundary of the symplectic area of $\Sigma(x, y, z)$ in lemma 4 below. In the case where $F = 0$, the family of reflective curves admits straight lines having endpoints $x(-t)$ and $x(t)$ and σ_x becomes the Grossmann–Royer transformation [29, 30]. The reflective curves are the natural generalization of the midpoint/chord construct introduced by Berezin, Berry and Marinov [11, 31, 32]. This generalization was used in [22] under the different name: symmetric curves.

3. Magnetic multiplication

Now we present formulae for the magnetic $*$ product. This product has two distinct representations. The first is given by a Berezin-type integral formula whose phase involves a three-sided symplectic area. The second is a left, right regular representation expressed in terms of pseudo-differential operators. We show how the family of magnetic reflections σ interrelates these different representations and how it is associated with the groupoid core of the $*$ product.

Observe that the linear envelope of quantizers forms an algebra. From (1.4) and (1.5) one has

$$\Delta(y)\Delta(x) = \int K(z, y, x)\Delta(z)\,dz. \tag{3.1}$$

The complex functions K are the symbols of the operators $\Delta(y)\Delta(x)$ and are regarded as the 'structure constants' of this algebra.

From lemma 1 (iii), it readily follows that

$$K(z, y, x) = (2\pi\hbar)^n \operatorname{tr}(\Delta(z)\Delta(y)\Delta(x)) = \frac{1}{(\pi\hbar)^{2n}} \exp\left\{\frac{i}{\hbar} \int_{M(z,y,x)} \omega\right\} \tag{3.2}$$

where $M(z, y, x)$ is a straight-line triangle having z, y, x as midpoints of its sides. The presence of the 'midpoint' delta functions in the quantizer kernel (2.7) collapses all the integrals in the trace giving the simple exponential result seen above. The symplectic membrane formula (3.2) in the magnetic phase space was first obtained (in another way) in [19]. A non-symplectic version of the three-magnetic quantizer trace can be found in [33].

The kernel K in (3.2) gives us the integral representation of the magnetic product (1.6),

$$(f * g)(z) = \iint K(z, y, x)f(y)g(x)\,dy\,dx. \tag{3.3}$$

The continuous function $K(z, y, x)$ is invariant under any cyclic permutation of arguments; under the permutation of any pair of arguments, $K \to \bar{K}$; it is the constant $(\pi\hbar)^{-2n}$ if any pair of arguments are the same. Also note that the trace operation in (3.2) is responsible for the gauge invariance of the $*$ product. Although Δ has a $U(1)$ gauge dependence, clearly M, ω and K are invariant.

It turns out that the boundary of the membrane $M(z, y, x)$ in (3.2) may be deformed in many different ways while leaving the kernel function $K(z, y, x)$ unchanged. Among these deformations is a representation stated in terms of the σ reflective curves which incorporates the groupoid properties of the magnetic product.

To see this, note that the magnetic product can be written

$$f * g = f(L)g = g(R)f \tag{3.4}$$

where $L = (L_q, L_p)$ and $R = (R_q, R_p)$ are Weyl-symmetrized sets of pseudo-differential operators on the phase space $\mathbb{R}^{2n}$. Since the map $f \to \hat{f}$ is Weyl ordered, $R = \bar{L}$. The associativity of $*$ implies $[L, R] = 0$.

In [19] it was established that

$$L_q = q + \tfrac{1}{2}i\hbar\partial_p \qquad\qquad R_q = q - \tfrac{1}{2}i\hbar\partial_p$$

$$L_p = p - \tfrac{1}{2}i\hbar\partial_q - A(L_q, R_q) \qquad R_p = p + \tfrac{1}{2}i\hbar\partial_q - A(R_q, L_q).$$

(3.5)

The vector function A is the two-point Valatin potential [34] (also referred to as the Schwinger–Fock or radial gauge potential in the literature), obeying

$$\frac{\partial A(q', q'')_j}{\partial q'^k} - \frac{\partial A(q', q'')_k}{\partial q'^j} = F_{jk}(q')$$

(3.6)

$$(q' - q'')^j A(q', q'')_j = 0.$$

(3.7)

So, with respect to its first argument, the potential $A(q', q'')$ represents a primitive of the Faraday 2-form, and satisfies the radial gauge condition (3.7). Taken together, equations (3.6) and (3.7) uniquely determine the potential $A(q', q'')$ giving the explicit formula

$$A(q', q'') = \int_0^1 F((\tau q' + (1 - \tau)q'')(q' - q'')\tau\, d\tau.$$

(3.8)

The Valatin potential is the τ-weighted average of the magnetic force on a unit charge moving with velocity $q' - q''$ from q'' to q'.

The function $\alpha_q(q')$, defined by (2.9) and used in our construction (2.11) of the magnetic reflections, is related to the Valatin potential via

$$\alpha_q(q') = A(q', 2q - q') + A(2q - q', q').$$

(3.9)

From this equality and from the explicit formulae (3.5) it is straightforward to determine the interrelationship between the L, R operators and the reflections σ. This statement requires the introduction of an extended phase space $T^*\mathbb{R}^{2n}$.

Lemma 3. *Definition (2.11) of the magnetic reflection is equivalent to the following operator identities:*

$$\sigma_x(R) = L.$$

(3.10)

Here L and R are the left and right operators (3.4) of the regular representation of the magnetic algebra. If L and R are represented by the symbols

$$L = l(x, -i\hbar\partial/\partial x) \qquad R = r(x, -i\hbar\partial/\partial x)$$

(3.11)

then (3.10) is equivalent to

$$\sigma_x(r(x, \eta)) = l(x, \eta) \qquad \eta \in T^*_x\mathbb{R}^{2n}$$

(3.12)

where

$$l_q(x, \eta) = x_q - \eta_p/2 \qquad\qquad l_p(x, \eta) = x_p + \eta_q/2 - A(l_q, r_q)$$

$$r_q(x, \eta) = x_q + \eta_p/2 \qquad\qquad r_p(x, \eta) = x_p - \eta_q/2 - A(r_q, l_q).$$

(3.13)

So the left and right vector functions l, r are defined on the space $T^*\mathbb{R}^{2n}$. Since the transformation $(x, \eta) \to (l, r)$ has a unique inverse either (x, η) or (l, r) may be used to represent points $m \in T^*\mathbb{R}^{2n} \approx \mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

Recall that groupoid multiplication is defined on the extended space in the following way. Two points $m_2, m_1 \in T^*\mathbb{R}^{2n}$ are multiplicable iff $r(m_2) = l(m_1)$, and in this case, their product is $m = m_2 \circ m_1$ where $l(m) = l(m_2)$ and $r(m) = r(m_1)$. The set of units e consists of the points where $r(m) = l(m)$ or $m = (x, y)$ with $y = 0$. The groupoid product

$\circ$ is non-commutative, associative and has inverse $(l, r)^{-1} = (r, l)$. The transformations $l : T^*\mathbb{R}^{2n} \to \mathbb{R}^{2n}, r : T^*\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are left and right (target and source) mappings of the groupoid structure on $T^*\mathbb{R}^{2n}$ which corresponds to the symplectic form ω on $\mathbb{R}^{2n}$ (see the general theory of symplectic groupoids in [20]). In view of (3.12), the magnetic reflections σ_x relate the left and right images $l(m)$ and $r(m)$ in the symplectic groupoid to each other via the central point $x = x(m)$.

A way to visualize how this groupoid structure can be used to construct the symplectic area phase for the $*$ product is the following. Given the three points $x_3, x_2, x_1 \in \mathbb{R}^{2n}$, solve the equation

$$m_3 \circ m_2 \circ m_1 = e \tag{3.14}$$

subject to the central conditions $x(m_i) = x_i = (q_i, p_i), i = 1, 2, 3$. It is easy to see that this problem has a unique solution. The q projected image of (3.14) is just the triangle $\delta(q_3, q_2, q_1)$ in $\mathbb{R}_q^n$ defined by its midpoints (q_3, q_2, q_1). The endpoints of sides of this triangle are the l_q, r_q values appearing in the Valatin potential, $A(l_q, r_q)$. Employing identities (3.13) fixes the three complementary endpoints l_p, r_p. Now consider a three-sided membrane $\Sigma(x_3, x_2, x_1)$ in $\mathbb{R}^{2n}$. Each side of its boundary is characterized by a triplet of points $[r_i, x_i, l_i]$ and some reflective curve that passes through these points. In the same way as in [19] one can check that

$$\int_{M(x_3, x_2, x_1)} \omega = \int_{\Sigma(x_3, x_2, x_1)} \omega. \tag{3.15}$$

This gives us a groupoid-consistent boundary for the $*$ product membrane. We note the allowed $\Sigma(x_3, x_2, x_1)$ boundary is non-unique or 'floppy' in character. Given the three sets of points $[l_i, x_i, r_i]$ satisfying the multiplicable property $l_i = r_{i+1}$ there are many reflective curves that are consistent with these data. Nevertheless the value $\int_{\Sigma(x_3, x_2, x_1)} \omega$ is the same for every allowed reflective curve boundary. As a result of (3.2) and (3.15) one has the groupoid-compatible form of K.

Lemma 4.

$$K(z, y, x) = \exp\left\{ \frac{i}{\hbar} \int_{\Sigma(z, y, x)} \omega \right\}. \tag{3.16}$$

Now using this formula for the kernel function K we obtain the associative non-commutative multiplication of functions on phase space by means of (1.8).

In order to specify the set of functions which is closed with respect to the operation (1.8) one needs to require smoothness and growth estimates on the magnetic field. We say that all the derivatives of the field tensor F have *polynomial growth at infinity* if for some $N < \infty$ there are estimates

$$\left| D_q^{\mathbf{s}} F_{jk}(q) \right| < C(\mathbf{s})(1 + |q|)^N \qquad C(\mathbf{s}) < \infty \qquad \mathbf{s} \in \mathbb{Z}_+^n. \tag{3.17}$$

The growth power N is independent of $\mathbf{s}$.

Theorem 1. *Let all the derivatives of the field tensor F have polynomial growth at infinity. Then the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ of all rapidly decreasing functions is closed with respect to the magnetic product (1.8). This associative algebra has the irreducible representation $f \to \hat{f}$ (1.4) in the Hilbert space $L^2(\mathbb{R}^n)$ so that relation (1.6) holds.*

The closure property is proved by using a representation of $f * g$ based on the Fourier-transformed symbols $\tilde{f}, \tilde{g}$ and suitable integration by parts of this representation. Property (1.6) is a consequence of (3.1). The irreducibility follows from the fact that the set of generators $\hat{q}, \hat{p}$ in (2.1) is irreducible.

The algebra $\mathcal{S}(\mathbb{R}^{2n})$ can be extended in order to include, say, polynomials in the generators $\hat{q}$, $\hat{p}$. But for this one has to place much stronger restrictions on the F tensor growth. If F has compact support one can certainly extend the algebra $\mathcal{S}(\mathbb{R}^{2n})$ to $\mathcal{S}^\infty(\mathbb{R}^{2n})$ consisting of smooth functions whose derivatives have polynomial growth at infinity. In this case the function $1 \in \mathcal{S}^\infty(\mathbb{R}^{2n})$ represents the unity element: $f * 1 = 1 * f = f$.

We note that, in view of lemma 1(iii), the magnetic Weyl correspondence $f \leftrightarrow \hat{f}$ is a unitary isomorphism from $L^2(\mathbb{R}^{2n})$ to the space of Hilbert–Schmidt operators in $L^2(\mathbb{R}^n)$. In this paper, we do not undertake a full investigation of the spaces of operators and symbols which realize the correspondence $f \leftrightarrow \hat{f}$. This is a separate technical (and often not simple) question which has been extensively studied in the pseudo-differential operator literature [13, 15, 20, 35–37]. The reader can consider all formulae as formally algebraic or, depending on the formula, assume an appropriate simple symbol class such as polynomials, smooth rapidly decreasing functions, etc.

4. Magnetic connection and $*$ product expansion

Let us now consider magnetic multiplication (1.8) as a one-parameter family of products depending on $\hbar$. Assume the smooth symbols f and g are $\hbar$ independent.

All the coefficients of the $\hbar \to 0$ expansion

$$f * g = fg + \sum_{k \geqslant 1} \frac{1}{k!}\left(-\frac{i\hbar}{2}\right)^k c_k(f, g) \tag{4.1}$$

were described in [19] in terms of partial derivatives of the functions f, g and of the magnetic tensor F (see also [38]).

Let us recall the structure of the first three coefficients $c_k(f, g)$. Of course, the leading term is the Poisson bracket corresponding to ω,

$$c_1(f, g) = \{f, g\} = f\langle \overleftarrow{D}\, \Psi\, \overrightarrow{D}\rangle g. \tag{4.2}$$

The next two terms are

$$c_2(\cdot, \cdot) = \langle \overleftarrow{D}\, \Psi\, \overrightarrow{D}\rangle^2 + \tfrac{2}{3}\partial_s F_{kl}(\overleftarrow{\partial^k}\, \overrightarrow{\partial^l}\, \overleftarrow{\partial^s} - \overleftarrow{\partial^k}\, \overrightarrow{\partial^l}\, \overrightarrow{\partial^s})$$

$$c_3(\cdot, \cdot) = \langle \overleftarrow{D}\, \Psi\, \overrightarrow{D}\rangle^3 + 2\langle \overleftarrow{D}\, \Psi\, \overrightarrow{D}\rangle \partial_s F_{kl}(\overleftarrow{\partial^k}\, \overrightarrow{\partial^l}\, \overleftarrow{\partial^s} - \overleftarrow{\partial^k}\, \overrightarrow{\partial^l}\, \overrightarrow{\partial^s}) \tag{4.3}$$

$$+ \partial^2_{sr} F_{kl}(\overleftarrow{\partial^k}\, \overrightarrow{\partial^l}\, \overleftarrow{\partial^s}\, \overleftarrow{\partial^r} + \overleftarrow{\partial^k}\, \overrightarrow{\partial^l}\, \overrightarrow{\partial^s}\, \overrightarrow{\partial^r}).$$

We indicate by $D = (\partial/\partial q, \partial/\partial p)$ the total derivative with respect to all the phase space variables and denote by $\partial^k = \partial/\partial p_k$ the derivative by the momentum component. The arrows indicate on which multiplier (left or right) the derivatives act. None of the derivatives act on the tensor components of Ψ or F. In particular

$$f(x)\langle \overleftarrow{D}(x)\, \Psi(x)\, \overrightarrow{D}(x)\rangle^N g(x) \equiv (D_{i_1}\cdots D_{i_N}f)(x)\Psi^{i_1 j_1}(x)\cdots \Psi^{i_N j_N}(x)(D_{j_1}\cdots D_{j_N}g)(x)$$

$$\partial^m_{s_1\cdots s_m} F_{kl}(q) \equiv \frac{\partial^m}{\partial q^{s_1}\cdots \partial q^{s_m}} F_{kl}(q).$$

The disadvantage of formulae such as (4.3) is that they are not in covariant form. In order to have a covariant expression one has to introduce a suitable connection on $\mathbb{R}^{2n}$ and replace the partial derivative D by a covariant derivative ∇. Thus the question at this stage is the following. What is the connection that is consistent with the magnetic quantization? Of course, the space $\mathbb{R}^{2n}$ is equipped with the trivial Euclidean connection. However, this Euclidean connection knows nothing about the magnetic 2-form ω and is not a symplectic connection.

But there is a natural symplectic connection induced by reflections σ.

Proposition 1. *The family of magnetic reflections $\{\sigma_x\}$ defines a symplectic connection with respect to ω via the Christoffel symbols (1.10). The components Γ^i_{jk} vanish unless $j, k \leqslant n$ and $i > n$. For $i, j, k \leqslant n$, the non-trivial portion of Γ at the point $x = (q, p) \in \mathbb{R}^{2n}$ is the following:*

$$\Gamma^{i+n}_{jk} \equiv \widetilde{\Gamma}_{ijk} = \frac{1}{2} \frac{\partial^2 \alpha_{q'}(q)_i}{\partial q^j \partial q^k}\bigg|_{q'=q}.$$

The explicit formula in terms of the magnetic tensor is

$$\widetilde{\Gamma}_{ijk} = \tfrac{1}{3}(\partial_k F_{ij} + \partial_j F_{ik}). \tag{4.4}$$

This connection has the curvature tensor

$$R^i_{jkl} = \frac{\partial}{\partial x^k} \Gamma^i_{jl} - \frac{\partial}{\partial x^l} \Gamma^i_{jk}$$

with the following non-zero components:

$$R^{i+n}_{jkl} \equiv \widetilde{R}_{ijkl} = \tfrac{1}{3}\partial^2_{ij} F_{kl}. \tag{4.5}$$

Proof. From (1.10), since the explicit form (2.11) of σ_x is known in terms of the Valatin potentials (see (2.10, 3.9)), the transformation of Γ under an arbitrary diffeomorphism $x \rightarrow \tilde{x} = \tilde{x}(x)$ is easily calculated and shows that Γ is a connection on $\mathbb{R}^{2n}$, cf [21]. The equality for the curvature results from the commutativity, $\Gamma^i_{sk}\Gamma^s_{jl} - \Gamma^i_{sl}\Gamma^s_{jk} = 0$.

Finally, consider the symplectic nature of this connection. Write the connection in block matrix form

$$\Gamma^i_{jk} = \Gamma[k]^i_j \qquad \Gamma[k] = \begin{pmatrix} 0 & 0 \\ \widetilde{\Gamma}[k] & 0 \end{pmatrix}. \tag{4.6}$$

In this notation the covariant derivative of ω takes the form

$$\nabla_k \omega_{ij} = (\partial_k \omega - \Gamma[k]^T \omega - \omega \Gamma[k])_{ij}.$$

One readily finds that

$$\nabla_k \omega = \begin{pmatrix} \partial_k F - \widetilde{\Gamma}[k] + \widetilde{\Gamma}[k]^T & 0 \\ 0 & 0 \end{pmatrix}.$$

All the terms in the matrix above are individually zero if $k > n$. For $k \leqslant n$ the upper left block is

$$\partial_k F_{ij} - \widetilde{\Gamma}[k]_{ij} + \widetilde{\Gamma}[k]_{ji} = \partial_k F_{ij} - \tfrac{1}{3}(\partial_k F_{ij} + \partial_j F_{ik}) + \tfrac{1}{3}(\partial_k F_{ji} + \partial_i F_{jk}) = 0.$$

The last equality is a consequence of the closedness of form (1.1). $\qquad \square$

If in $\widetilde{\Gamma}_{ijk}$ the factor $1/3$ is replaced by any other number then Γ ceases to be a symplectic connection. Also, observe that Γ is a function of q, but not of p.

Remark. As is evident from its construction the connection Γ on $\mathbb{R}^{2n} = \mathbb{R}^n_q \oplus \mathbb{R}^n_q$ is torsion free and non-metrical. However, it does depend on the metric in the following way. The construction (2.11) of the q, p-components of the reflection map employs Euclidean geodesics. The q linearity of q-components and the p linearity of the p-components of σ_x are responsible for the 0-blocks in the tensor structure of Γ and R.

Stated covariantly formula (4.4) reads

$$\widetilde{\Gamma}_{ijk} = \tfrac{1}{3}\left(\nabla^0_k F_{ij} + \nabla^0_j F_{ik}\right) \tag{4.7}$$

where ∇_j^0 are covariant derivatives on configuration space. In the framework of the present paper $\nabla_j^0 = \partial_j$ are just the Euclidean derivatives on $\mathbb{R}_q^n$, but we can claim that formula (4.7) actually represents the magnetic contribution to the phase space connection also on $T^*\mathcal{M}$ for any affine (in particular, Riemannian) configuration manifold $\mathcal{M}$ where ∇^0 is non-trivial, for instance, non-flat. In the general case, of course, the three other blocks in (4.6) will no longer be zero (compare with [6]). We postpone the corresponding details to another paper.

Proposition 2. *The coefficients c_2 and c_3 in the $*$ product expansion (4.1) have the covariant form*

$$c_2(\cdot, \cdot) = \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^2 \tag{4.8}$$

$$c_3(\cdot, \cdot) = \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^3 + R^{ijkl}(\overrightarrow{\nabla}_i \overrightarrow{\nabla}_j \overrightarrow{\nabla}_k \overleftarrow{\nabla}_l - \overleftarrow{\nabla}_i \overleftarrow{\nabla}_j \overleftarrow{\nabla}_k \overrightarrow{\nabla}_l). \tag{4.9}$$

Here $R^{ijkl} = R^i_{\ j'k'l'}\Psi^{j'j}\Psi^{k'k}\Psi^{l'l}$, and ∇ is the covariant derivative with respect to the symplectic connection Γ (1.10).

Proof. The non-covariant form of the $k = 2$ coefficient reads
$$c_2(f, g) = f \langle \overleftarrow{D} \Psi \overrightarrow{D} \rangle^2 g + f M_0(\overleftarrow{\partial}, \overrightarrow{\partial}) g.$$
The M_0 factor above is
$$M_0(u, v) = \tfrac{2}{3} u^k u^s \partial_s F_{kl} v^l - \tfrac{2}{3} u^k \partial_s F_{kl} v^l v^s = -(u^k u^s \widetilde{\Gamma}_{lsk}(q) v^l + u^k \widetilde{\Gamma}_{kls}(q) v^l v^s).$$
Straightforward calculations show that
$$\langle \overleftarrow{D} \Psi \overrightarrow{D} \rangle^2 + M_0(\overleftarrow{\partial}, \overrightarrow{\partial}) = \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^2.$$
This establishes (4.8).

Consider next the $O(\hbar^3)$ coefficient. It has the non-covariant form
$$c_3(f, g) = f(\langle \overleftarrow{D} \Psi \overrightarrow{D} \rangle^3 + 3\langle \overleftarrow{D} \Psi \overrightarrow{D} \rangle M_0(\overleftarrow{\partial}, \overrightarrow{\partial}) + 3 M_1(\overleftarrow{\partial}, \overrightarrow{\partial})) g$$
where
$$M_1(u, v) = \tfrac{1}{3}(u^k u^s u^r \partial_{sr}^2 F_{kl} v^l + u^k \partial_{sr}^2 F_{kl} v^l v^s v^r) = \widetilde{R}_{srkl}(u^s u^r u^k v^l - v^s v^r v^k u^l).$$
A tensor computation then shows that
$$\langle \overleftarrow{D} \Psi \overrightarrow{D} \rangle^3 + 3\langle \overleftarrow{D} \Psi \overrightarrow{D} \rangle M_0(\overleftarrow{\partial}, \overrightarrow{\partial}) + 2 M_1(\overleftarrow{\partial}, \overrightarrow{\partial}) = \langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^3.$$
Thus the $O(\hbar^3)$ coefficient becomes
$$c_3(f, g) = f(\langle \overleftarrow{\nabla} \Psi \overrightarrow{\nabla} \rangle^3 + \widetilde{R}_{srkl}(\overleftarrow{\partial}^s \overleftarrow{\partial}^r \overleftarrow{\partial}^k \overrightarrow{\partial}^l - \overrightarrow{\partial}^s \overrightarrow{\partial}^r \overrightarrow{\partial}^k \overleftarrow{\partial}^l)) g.$$
Finally, index raising by $\Psi(q)$ on $\widetilde{R}$ gives the covariant $c_3(f, g)$ expression in (4.9). $\square$

5. Quantization with arbitrary electromagnetic fields

So far it has been assumed that the magnetic fields are static. Now we consider the modifications in the Weyl quantization that arise when the electromagnetic fields are time dependent. The manner in which the electric field enters the symbol calculus is made explicit.

First it is helpful to clarify the role of the vector potential A in the static Weyl quantization. Within the quantum phase space framework, the potential A never appears. The ω symplectic form and Poisson brackets, the symplectic area Σ, the connection Γ, the $*$ product and its expansion coefficients c_k are all defined directly in terms of the magnetic tensor F. However,

the 2-point Valatin potential $A(q', q'')$ (3.8), which is a non-local gauge invariant object, does appear spontaneously as a contribution to the l, r functions and is essential in the definition of the reflection symmetry σ_x and the groupoid product. Only when one goes to the Hilbert space representation, via $f \to \hat{f}$, cf (1.6), (2.1), is any gauge fixing required. In order to represent the quantizer $\Delta(x)$, a vector potential A (consistent with F) must be employed. One convenient gauge choice for A is to use again the Valatin potential with a fixed 2nd argument. This was the option selected in our prior work [19], where the fixed 2nd argument was set to 0.

Let $B(t, q)$ denote the magnetic field. In the $n = 3$, time-dependent case the 2-form becomes

$$\omega(t) = \mathrm{d}p \wedge \mathrm{d}q + B_1(t, q)\,\mathrm{d}q^2 \wedge \mathrm{d}q^3 + B_2(t, q)\,\mathrm{d}q^3 \wedge \mathrm{d}q^1 + B_3(t, q)\,\mathrm{d}q^1 \wedge \mathrm{d}q^2. \tag{5.1}$$

Similarly, the quantum commutation relations (1.2) acquire time dependence via the momentum components by $[\hat{p}_j(t), \hat{p}_k(t)] = \mathrm{i}\hbar\epsilon_{jkl}B_l(t, \hat{q})$.

Replacing the static ω with $\omega(t)$ in (1.8) defines a $*$ product that is time dependent. This time dependence results from the t varying magnetic flux through the triangle $\delta(q_3, q_2, q_1)$. Likewise the quantizer, the left, right coordinates, the reflection symmetry σ_x and the magnetic connection all acquire an obvious t dependence.

In order to fix the quantizer and obtain a unique irreducible Hilbert space representation, the quantum coordinates (2.1) need to be defined. Let us work in the Coulomb gauge where the 4-vector potential $a(t, q) \equiv \{-\phi(t, q), A(t, q)\}$ has a vanishing scalar component, $\phi(t, q) = 0$. There is no loss of generality in this Coulomb gauge assumption since given a general 4-vector one may, by a known unitary transformation, always gauge away the scalar component. In the Coulomb gauge

$$B(t, q) = \nabla \times A(t, q) \qquad E(t, q) = -\frac{\partial}{\partial t}A(t, q). \tag{5.2}$$

Define $\hat{q}$, $\hat{p}(t)$ by (2.1) with the static $A(q)$ replaced by $A(t, q)$. From (2.1) and (5.2) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}(t) = E(t, \hat{q}). \tag{5.3}$$

This approach is based on the separation of time and space variables that is needed for solving the Cauchy problem, see details in [19]. With this separation we observe that the quantum phase space coordinates have acquired time-dependent momentum components. The magnetic field $B(t, q)$ determines the symplectic structure via (5.1), whereas in the Coulomb gauge the electric field $E(t, q)$ generates the motion of the kinetic coordinates $\hat{p}(t)$. The magnetic curvature $R^i_{jkl}(t, q)$ is time dependent on the phase space $\mathbb{R}^6 = \mathbb{R}^3_q \oplus \mathbb{R}^3_p$ and does not sense the electric field.

Another view point is to include the time and space variables together into the configuration space $\mathbb{R}^4_{t,q}$. Then the symplectic form and the magnetic curvature tensor on the space $\mathbb{R}^8 = \mathbb{R}^4_{t,q} \oplus \mathbb{R}^4_{p_t,p}$ will now depend on the electric field as well.

6. Zero magnetic curvature

Let us return to the static situation. The magnetic connection (1.10) is determined by the first derivatives of the tensor F. Its curvature is determined by the second derivatives.

The simplest case is the *homogeneous magnetic field* that is $F = $ const. In this case the Christoffel symbols Γ are just zero (in the Euclidean basis), and the magnetic connection coincides with the Euclidean connection on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$.

The second simple example is that of a *linear magnetic field*, that is

$$F_{ij}(q) = F_{ij,k}q^k \qquad F_{ij,k} = -F_{ji,k} \qquad F_{ij,k} + F_{jk,i} + F_{ki,j} = 0. \qquad (6.1)$$

This is the Lie algebra case; the commutation relations (1.2) are linear. In this case the magnetic connection becomes a constant,

$$\widetilde{\Gamma}_{ijk} = \tfrac{1}{3}(F_{ij,k} + F_{ik,j}) \qquad (6.2)$$

and so the magnetic curvature is zero: $R = 0$. The reflection symmetry σ_x in this case is realized by quadratic mappings

$$\sigma_x(y) = 2x - y - \begin{pmatrix} 0 \\ \widetilde{\Gamma}_{\cdot,jk}(x_q - y_q)^j(x_q - y_q)^k \end{pmatrix}. \qquad (6.3)$$

Let us consider the following non-symplectic change of variables in $\mathbb{R}^{2n}$:

$$q' = q \qquad p' = p + A(q). \qquad (6.4)$$

Here A is a magnetic potential satisfying (2.2). Under this transformation the magnetic form ω is transformed into the canonical form: $\omega' = \mathrm{d}p' \wedge \mathrm{d}q'$.

For instance, one can take A to be the Valatin potential with fixed second argument 0,

$$A_i(q) = A_i(q, 0) = \tfrac{1}{3}F_{ij,k}q^j q^k = \tfrac{1}{2}\widetilde{\Gamma}_{ijk}q^j q^k \qquad (6.5)$$

which satisfies the radial gauge condition (3.7), $q^i A_i(q) = 0$. For a quadratic A, the $x \to x'$ variable change maps the magnetic connection Γ into $\Gamma' = 0$ (the Euclidean connection).

The form ω' and the connection Γ' generate the usual Groenewold $*$ product over $\mathbb{R}^{2n}$, which can be expressed both in the integral form and in the derivative form

$$(f' *' g')(x') = \frac{1}{(\pi\hbar)^{2n}} \iint \exp\left\{ \frac{i}{\hbar} \int_{M(x',y'z')} \omega' \right\} f'(y')g'(z') \, \mathrm{d}y' \, \mathrm{d}z' \qquad (6.6)$$

$$(f' *' g')(x') = f'(x') \exp\left\{ -\frac{i\hbar}{2} \langle \overleftarrow{D}' \Psi' \overrightarrow{D}' \rangle \right\} g'(x'). \qquad (6.7)$$

Here $\Psi' = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ is the Poisson tensor corresponding to the symplectic form ω', the derivatives $D' = \partial/\partial x'$ are taken with respect to the coordinates $x' = (q', p')$, and the membrane $M(x', y', z')$ is just the triangle in $\mathbb{R}^{2n}$ with midpoints $x', y'z'$.

In formulae (6.7), (6.6) we denote the symbols f', g' by primes in orders to emphasize that these functions are expressed in the new coordinate system $x' = (q', p')$. Of course, there is a correspondence with the functions in the previous (magnetic) coordinate system $x = (q, p)$, namely

$$f(x) = f'(x') \qquad x \leftrightarrow x' \quad \text{by} \quad (6.4).$$

In the magnetic coordinates we have the magnetic product (1.8). So the question arises: does this magnetic product correspond to the Groenewold product under this change of variables?

Proposition 3. *Assume that the magnetic curvature R is zero, that is the magnetic tensor F is linear. Let the change of variables (6.4) satisfy the radial gauge condition, i.e. the potential A is given by (6.5). Then under this quadratic change of variables $x \leftrightarrow x'$ the magnetic product (1.8) generated by the form ω (1.1) corresponds to the Groenewold product (6.6) generated by the form $\omega' = \mathrm{d}p' \wedge \mathrm{d}q'$.*

In particular, the Groenewold differential formula (6.7) implies the following representation of the magnetic product:

$$(f * g)(x) = f(x) \exp\left\{ -\frac{i\hbar}{2} \overleftarrow{\nabla} \Psi(x) \overrightarrow{\nabla} \right\} g(x) \qquad (6.8)$$

where Ψ is the magnetic Poisson tensor, and where the covariant derivatives are given by the flat magnetic connection (6.2).

Proof. First perform the variable change $x \to x'$ in the M-integral representation of the $*$ product, cf (3.2), (3.3). If A is quadratic, one readily finds that

$$\iint \exp\left\{\frac{i}{\hbar}\int_{M(z,y,x)}\omega\right\} f(y)g(x)\,\mathrm{d}y\,\mathrm{d}x = \iint \exp\left\{\frac{i}{\hbar}\int_{M(z'(z),y',x')}\omega'\right\} f'(y')g'(x')\,\mathrm{d}y'\,\mathrm{d}x'.$$

This establishes that $(f * g)(x) = (f' *' g')(x'(x))$ and verifies that $f * g$ is given by the right-hand side of (6.7). Now implement the inverse transform $x' \to x$ and employ $f'\langle \overleftarrow{D}' \Psi' \overrightarrow{D}'\rangle^N g' = f\langle \overleftarrow{\nabla} \Psi(x) \overrightarrow{\nabla}\rangle^N g$ to obtain (6.8). $\qquad\square$

The content of proposition 3 agrees well with known formulae for formal $*$ products over flat symplectic manifolds [1, 12]. But we see that our formula (6.8) actually holds for the non-formal strict $*$ product (1.8) which has the operator representation (1.6), and that the connection ∇ in (6.8) is exactly the magnetic connection (1.10).

The change of variables (6.4) can also be carried out for general nonlinear tensors F. Again the Groenewold formula (6.7) generates in this way a certain product

$$(f \times g)(x) \equiv f(x)\exp\left\{-\frac{i\hbar}{2}\langle \overleftarrow{\nabla}^A \Psi(x) \overrightarrow{\nabla}^A\rangle\right\} g(x) \tag{6.9}$$

where ∇^A corresponds to the flat connection with Christoffel symbol components $\widetilde{\Gamma}_{ijk} = \partial^2_{jk} A_i(q)$. However, here the approach of deriving the magnetic $*$ product through the variable change (6.4) fails. The product (6.9) is not related to the magnetic product and the flat connection ∇^A is not the magnetic connection, if F is not linear.

7. Constant magnetic curvature

Constant magnetic curvature means that the tensor F is quadratic. There is no loss in generality here in assuming that F is purely quadratic with no linear component. In detail

$$F_{ij}(q) = F_{ij,kl}q^k q^l \qquad F_{ij,kl} = -F_{ji,kl} \qquad F_{ij,kl} = F_{ij,lk}$$
$$F_{ij,kl} + F_{jk,il} + F_{ki,jl} = 0. \tag{7.1}$$

In this case, the commutation relations (1.2) are quadratic and the algebra is not a Lie algebra.

The magnetic connection and curvature are given by

$$\widetilde{\Gamma}_{ijk}(q) = \tfrac{2}{3}(F_{ij,kl} + F_{ik,jl})q^l \qquad \widetilde{R}_{ijkl} = \tfrac{2}{3}F_{kl,ij}. \tag{7.2}$$

The magnetic reflection is still a quadratic mapping:

$$\sigma_x(y) = 2x - y - \begin{pmatrix} 0 \\ \widetilde{\Gamma}(x_q)_{\cdot,jk}(x_q - y_q)^j(x_q - y_q)^k \end{pmatrix}. \tag{7.3}$$

Lemma 5. *Let the Faraday F tensor be quadratic, that is, the magnetic curvature be constant. Then the reflections σ_x (7.3)*

(i) are affine with respect to the magnetic connection (map geodesics into geodesics);
(ii) satisfy the symmetry condition (1.12) and
(iii) coincide with geodesic reflections generated by the magnetic connection.

Proof.   (i) Let $\gamma(\xi) = (q(\xi), p(\xi)), \xi \in [-1, 1]$ be a generic magnetic geodesic. The zero block structure of Γ allows one to state the geodesic equation of motion as

$$\ddot{\gamma}^i(\xi) + \Gamma^i_{jk}(q(\xi))\dot{q}^j(\xi)\dot{q}^k(\xi) = 0 \qquad j, k \leqslant n. \tag{7.4}$$

Since $\Gamma^i_{jk} = 0$ for $i \leqslant n$, $\ddot{q}(\xi) = 0$ and $\dot{q}(\xi) = \text{const.}$

Set $\gamma'(\xi) \equiv \sigma_x(\gamma(\xi))$; we must show γ' is a geodesic. For σ_x given by (7.3), the second derivative of γ' is

$$\ddot{\gamma}'^i(\xi) = -\ddot{\gamma}^i(\xi) - 2\Gamma^i_{jk}(x_q)\dot{q}^j\dot{q}^k.$$

Use $q(\xi) + q'(\xi) = 2x_q$; the q-linearity of Γ, $2\Gamma^i_{jk}(x_q) = \Gamma^i_{jk}(q(\xi)) + \Gamma^i_{jk}(q'(\xi))$ and $\dot{q} = -\dot{q}'$ to show the above identity is equivalent to

$$\ddot{\gamma}'^i(\xi) + \Gamma^i_{jk}(q'(\xi))\dot{q}'^j\dot{q}'^k = -\big(\ddot{\gamma}^i(\xi) + \Gamma^i_{jk}(q(\xi))\dot{q}^j\dot{q}^k\big) = 0. \tag{7.5}$$

Thus γ' is a geodesic.

A similar argument verifies (iii); item (ii) results from a straightforward algebraic calculation.   □

Note that property (ii) means that in the quadratic case the reflections σ_x (7.3) determine the symmetric symplectic structure on the phase space $\mathbb{R}^{2n}$ in the sense of [23].

Corollary 1. *If the magnetic tensor is quadratic, then the magnetic product can be represented by formula (1.8) using membranes $\Sigma(z, y, x)$ bounded by three magnetic geodesics with midpoints z, y, x.*

Considering higher terms in the formal $\hbar$-power expansion (4.1) in the covariant form (1.9), (4.8) and (4.9) we can conjecture the following generalization of the Groenewold representation.

Conjecture. *If the magnetic curvature is constant (that is, F is quadratic), then*

$$f * g = f \exp\left\{-\frac{i\hbar}{2}\langle \overleftarrow{\nabla} \Psi(x) \overrightarrow{\nabla}\rangle + \frac{i\hbar^3}{48} R^{ijkl}(\overrightarrow{\nabla}_i \overrightarrow{\nabla}_j \overrightarrow{\nabla}_k \overleftarrow{\nabla}_l - \overleftarrow{\nabla}_i \overleftarrow{\nabla}_j \overleftarrow{\nabla}_k \overrightarrow{\nabla}_l)\right\} g \tag{7.6}$$

where R is the magnetic curvature with raised indices, and ∇ is the magnetic connection.

8. Conclusions

The quantum coordinates $\hat{q}^j$ and kinetic momenta $\hat{p}_k$ of a charged particle know about the presence (or absence) of magnetic field via the commutation relations between momenta [39]. In general, these commutation relations are nonlinear.

Weyl-symmetrized functions in the operators $\hat{q}, \hat{p}$ form an algebra. The symbol image of the multiplication in this magnetic algebra can be represented (exactly) by a simple integral formula (1.8) via the magnetic symplectic form ω and membranes Σ having a groupoid-consistent boundary. This is an example of perfectly quantizable phase space.

The groupoid structure, corresponding to the form ω, is controlled by a family of magnetic reflections (2.11), which are generated by the regular left and right representations (3.5) of the magnetic algebra.

The family of magnetic reflections determines a symplectic connection Γ, (1.10). The magnetic $*$ product (1.8) has a covariant derivative asymptotic expansion (1.9) whose $\hbar^3$ term (1.11) is given by the curvature of this connection.

In the zero-curvature case, the magnetic $*$ product can be independently recovered from the standard (non-magnetic) Groenewold exponential formula by a non-symplectic change

of variables. The resultant exponential formula is stated in terms of covariant derivatives generated by the magnetic connection.

The case of constant (but non-zero) curvature represents an interesting example of a symplectic symmetric space. The magnetic $*$ product in this case is given either via a geodesic bounded membrane area or via the explicit covariant differential formula (7.6) with magnetic curvature tensor in the exponent.

The $\hbar^3$ term in the $\hbar \to 0$ expansion of our magnetic $*$ product, in the constant-curvature case, is different from the corresponding term in the known Bieliavsky–Cahen–Gutt product [23] (given for general symplectic symmetric spaces). In the general non-constant-curvature case a difference of numerical coefficient in the $\hbar^3$ terms can also be observed in the comparison with the Fedosov deformation expansion [4].

Of course, on the level of formal $\hbar$ power series all associative $*$ products are equivalent [5]. But only some of them, such as our expansions (1.9), (1.11), are related to exact products possessing an irreducible operator representation in a Hilbert space. Such an operator representation does not allow generic $\hbar$-pseudo-differential transformations (allowed by the formal $*$ products). The condition that the $*$ product admits an exact irreducible operator representation in essence restricts the variety of $*$ products (see in [7]).

The magnetic connection and its curvature, which we extract from the quantum algebra, is determined by the magnetic tensor F, but not in the way it usually appears in gauge-field theory via $U(1)$-line bundles [14], nor in the way suggested by Weyl nearly a century ago [40]. For instance, the magnetic connection is defined on the phase space rather than on the configuration space. This magnetic connection is clearly important in the quantization process, but we also anticipate that it can be observed in the dynamical and spectral problems involving magnetic fields such as the Fock–Landau level problem in the inhomogeneous case.

Acknowledgments

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